Nour Bachir El-Bayadh University Center, El-Bayadh, Algeria

1st year LMD Mathematics and computer science

Analysis 1

TD Session 1

Exercise 1

1. Find which of the variables x, y, z and u represent rational numbers and which irrational numbers: (i) $x^2 = 7$, (ii) $y^2 = 16$, (iii) $z^2 = 0.036$ and (iv) $u^2 = \frac{3}{4}$.
2. Show that $5 + \sqrt{2}$ is irrational.
3. If $a = 3 + 2\sqrt{6}$ and $b = \frac{1}{a}$, what will be the value of $a^2 + b^2$.

Exercise 3

Solve in $\mathbb{Z}$, respectively in $\mathbb{Z}^2$, the equations:

$$x^3 - 5x^2 + 8 = 0 \text{ and } 2x - 3y = 5.$$

Exercise 4

Determine, if they exist, the lower and upper limits of the following sets (discuss according to the parameter values):

$$A = \{(-1)^k + \tfrac{(-1)^n}{n+1} / k, n \in \mathbb{N}\},\ B_{\alpha,\beta} = \{x^2 - x + 1 / x \in [\alpha, \beta]\}.$$

Exercise 5

Solve in $\mathbb{R}$ the inequality:

$$|2x^2 - 1| \leq |x + 1|.$$

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Analysis 1

TD Session 2

Exercise 1

Show that $\sqrt{2}$, $\frac{\ln(3)}{\ln(2)}$ and π are irrational numbers.

Exercise 2

Determine the least upper and greatest lower bound (if they exist) of $A = \{u_n : n \in \mathbb{N}\}$ with

$$u_n = \begin{cases} 2^n, & \text{if } n \text{ is even} \\ 2^{-n}, & \text{if } n \text{ is odd.} \end{cases}$$

Exercise 3

Establish the following results ($[x]$ denotes the integer part of x):

1. $\forall x, y \in \mathbb{R}$, $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$.
2. $[x] + [-x] = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ -1, & \text{if } x \notin \mathbb{Z}. \end{cases}$

Exercise 4

Let A and B be two bounded parts of $\mathbb{R}$. True or false?

1. $A \subset B \Rightarrow \sup A \leq \sup B$,
2. $A \subset B \Rightarrow \inf A \leq \inf B$,
3. $\sup(A \cup B) = \max\{\sup A, \sup B\}$,
4. $\sup(A + B) < \sup A + \sup B$,
5. $\sup(-A) = -\inf(A)$,
6. $\sup A + \inf B \leq \sup(A + B)$.

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Analysis 1

TD Session 3

Exercise 1

1. Verify that

   (a) $\sqrt{2} - i - i(1 - \sqrt{2}i) = -2i$ and (b) $(2 - 3i)(-2 + i) = -1 + 8i$.

2. Reduce the quantity
$$\frac{5i}{(1-i)(2-i)(3-i)}$$
   to a real number.

3. Show that (a) $Re(iz) = -Im(z)$ and (b) $Im(iz) = Re(z)$.

Exercise 2

1. show that $(z_1z_2)z_3 = z_1(z_2z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

2. Compute (a) $\frac{2+i}{2-i}$ and (b) $(1 - 2i)^4$.

3. Let f be the map sending each complex number
$$z = x + iy \to \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$
   Show that $f(z_1z_2) = f(z_1)f(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

4. Use the binomial theorem to expand

   (a) $(1 + \sqrt{3}i)^{2011}$ and (b) $(1 + \sqrt{3}i)^{-2011}$.

Exercise 3

Graph the following regions in the complex plane:

(a) $\{z : Re(z) \geq 2Im(Z)\}$,

(b) $\{z : \frac{\pi}{2} \leq Arg(z) \leq \frac{3\pi}{4}\}$,

(c) $\{z : |z - 4i + 2| > 2\}$.

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Analysis 1

TD Session 4

Exercise 1

1. Find all complex solutions of the following equations:

   (a) $\bar{z} = z$, (b) $\bar{z} + z = 0$, (c) $\bar{z} = \frac{9}{z}$.

2. Suppose that z_1 and z_2 are complex numbers, with $z_1 z_2$ real and non-zero. Show that there exists a real number r such that $z_1 = r\bar{z}_2$.

Exercise 2

1. show that
   $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ for all $z_1, z_2 \in \mathbb{C}$.

2. Verify that $\sqrt{2}|z| \geq |Re(z)| + |Im(z)|$.

3. Sketch the curves in the complex plane given by

   (a) $Im(z) = -1$, (b) $|z - 1| = |z + i|$, (c) $2|z| = |z - 2|$.

Exercise 3

1. Express the following in the form $x + iy$ with $x, y \in \mathbb{R}$

   (a) $\frac{i}{1-i} + \frac{1-i}{i}$, (b) all the 3rd roots of $-8i$, (c) $(\frac{i+1}{\sqrt{2}})^{1337}$.

2. Find the principal argument and exponential form of

   (a) $z = \frac{i}{1+i}$, (b) $z = \sqrt{3} + i$, (c) $z = 2 - i$.

3. Find all the complex roots of the equations:
   (a) $z^6 = -9$, (b) $z^2 + z + (1 - i) = 0$.

4. Find the four roots of the polynomial $z^4 + 16$ and use these to factor $z^4 + 16$ into two quadratic polynomials with real coefficients.

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Analysis 1

TD Session 5

Exercise 1

(A) Using the definition of the limit of a sequence, demonstrate that each of the sequences (u_n) converges to the limit l:
(1) $u_n = \frac{1}{n}, l = 0$, (2) $u_n = \frac{(-1)^{n+1}}{n}, l = 0$, (3) $u_n = \frac{4n-1}{2n+1}, l = 2$.

(B) Let $(u_n)_{n\geq 0}$ be a sequence of non-zero real numbers such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0$. Show that $(u_n)_{n\geq 0}$ converges to 0.

Exercise 2

Let a be a real number. We define the real sequence $(u_n)_{n\geq 0}$ by

$$\begin{cases} u_0 = a \\ u_{n+1} = u_n^2 + \frac{1}{4}. \end{cases}$$

Show that $(u_n)_{n\geq 0}$ is increasing. assuming that $(u_n)_{n\geq 0}$ is bounded above, determine the (possible) limit. Finally, discuss the existence of the limit according to the values of a.

Exercise 3

We define for $n \geq 1$ the two sequences

$$u_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n+1}, \quad v_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n}.$$

Show that these two sequences are adjacent. Deduce the value of the limit $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$.

Exercise 4

Show that by induction on p that $\forall n, p \in \mathbb{N}^*$, $\frac{1}{(n+1)^2} + ... + \frac{1}{(n+p)^2} < \frac{1}{n} - \frac{1}{n+p+1}$. Deduce that the sequence defined by $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is a Cauchy sequence.

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Analysis 1

TD Session 6

1. Determine domains of the following real functions:
   $f(x) = \sqrt{\frac{x^2-1}{x^3-1}}, \quad g(x) = \ln(1 - 2\cos x), \quad h(x) = \ln(\frac{2-|x|}{|x|-1})$.

2. Determine for $x \in \mathbb{R}$, whether the function f is continuous at l:
   (a) $f(x) = x^3 - 2x^2$, $l = 2$, (b) $f(x) = [x]$, $l = 1$, (c) $l = 0$, $f(x) = \begin{cases} x\sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$
   (d) $l = 0,\ f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$

3. Prove that the functions $f(x) = 1$ and $g(x) = x^{\frac{1}{n}}$ are continuous on $\mathbb{R}$.

4. Determine the points of continuity and discontinuity of the signum function

$$f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

5. Let the function $f : [0,1] \to [0,1]$ be continuous. Prove that there is a real number c in $[0,1]$ such that $f(c) = c$.

6. Determine whether the following limits exist:
   (a) $\lim\limits_{x\to 0} \dfrac{x^2+x}{x}$, (b) $\lim\limits_{x\to 0} \dfrac{|x|}{x}$, (c) $\lim\limits_{x\to 0} \dfrac{\sin x}{2x+x^2}$, (d) $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x^2}$, (e) $\lim\limits_{x\to 0} (\dfrac{x}{\sin x})^{\frac{1}{2}}$,
   (f) $\lim\limits_{x\to 0} \dfrac{1}{|x|}$, (g) $\lim\limits_{x\to 1^+} \dfrac{1}{1-x^3}$,

7. For the functions $f(x) = \begin{cases} 0, & x = 0, \\ -2, & x = 1, \\ 2+x, & x \neq 0, 1, \end{cases}$ and $g(x) = \begin{cases} 0, & x = 0, \\ 1+x, & x \neq 0, \end{cases}$

   determine $(f \circ g)(x)$, $(f \circ g)(0)$, $f(\lim\limits_{x\to 0} g(x))$ and $\lim\limits_{x\to 0} f(g(x))$.

8. Prove that the function $f(x) = x^2$ is uniformly continuous on $[2, 3]$.

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Analysis 1

TD Session 7

1. Prove that the function $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, is differentiable at any point c, and determine $f'(c)$.

2. Prove that the constant function on $\mathbb{R}$ is differentiable at any point c, with derivative zero.

3. Prove that the function $f(x) = \frac{1}{x}$, $x \in \mathbb{R} - \{0\}$ is differentiable at any point $c \neq 0$, and determine $f'(c)$.

4. Use the $\varepsilon - \delta$ definition of differentiability to prove that the function $f(x) = x^4$ is differentiable at 1, with derivative $f'(1) = 1$.

5. Prove that the following functions f are not differentiable at the given point c with $x \in \mathbb{R}$: (a) $f(x) = |x|^{\frac{1}{2}}$, $c = 0$, (b) $f(x) = [x]$, $c = 1$.

6. Prove that the function $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$ is differentiable at 0. Given that, for $x \neq 0$, $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$, is f' continuous at 0?.

7. Prove that the function $f(x) = \begin{cases} x^2, & x < 0, \\ x^3, & x \geq 0. \end{cases}$ is differentiable on $\mathbb{R}$. Is f' differentiable at 0?

8. Find the third order derivative of the function $f(x) = xe^{2x}$, $x \in \mathbb{R}$.

9. For each of the following functions f, show that f^{-1} is differentiable and determine its derivative: (a) $f(x) = \cos x$, $x(0, \pi)$ (b) $f(x) = \sinh x$, $x \in \mathbb{R}$.

10. Find the local extrema of the function $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3$, $x \in [-1, 2]$.

11. Use the above Strategy to determine the minimum and the maximum of the function $f(x) = \sin^2 x + \cos x$, for $x \in [0, \frac{\pi}{2}]$.

12. Verify that the conditions of Rolle's Theorem are satisfied by the function $f(x) = x^4 - 4x^3 + 3x^2 + 2$, $x \in [1, 3]$ and determine a value of c in $(1, 3)$ for which $f'(c) = 0$.

13. For each of the following functions, verify that the conditions of the Mean Value Theorem are satisfied, and find a value for c that satisfies the conclusion of the theorem: (a) $f(x) = x^3 + 2x$, $x \in [-2, 2]$, (b) $f(x) = e^x$, $x \in [0, 3]$.

14. Determine the tangent approximation to each of the following functions f at the given point a: (a) $f(x) = e^x$, $a = 2$; (b) $f(x) = \cos x$, $a = 0$.

15. Obtain an expression for $R_1(x)$ when Taylor's Theorem is applied to the function $f(x) = \frac{1}{1-x}$ at $a = 0$. Calculate the value of c when $x = \frac{3}{4}$.

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Analysis 1

TD Session 8

1. Study the function $f(x) = \ln(x^2 - \sqrt{x^2-1})$. Draw its representation curve.
2. Prove that $\frac{e^x}{x^n} \to \infty$ as $x \to \infty$, for each $n = 0, 1, 2, ...$
3. Prove that $e^x > x^e$, for $x > e$.
4. show that
$$1 + \cosh x + \cosh 2x + ... + \cosh nx = \frac{1}{2} + \frac{\cosh nx - \cosh(n+1)x}{2(1-\cosh x)}.$$
5. Simplify the expression $\frac{2\cosh^2 x - \sinh(2x)}{x - \ln(\cosh) - \ln 2}$ and give its limits as $x \to -\infty$ and $x \to \infty$.
6. Prove that $\sinh^{-1} x = \log_e(x + \sqrt{x^2+1})$, for $x \in \mathbb{R}$.
7. Prove that $\cosh^{-1} x = \log_e(x + \sqrt{x^2-1})$, for $x \in [1, \infty)$.